Stochastic and BOLTZMANN-like models for behavioral changes, and their relation to game theory

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Abstract

In the last decade, stochastic models have shown to be very useful for quantitative modelling of social processes. Here, a configurational master equation for the description of behavioral changes by pair interactions of individuals is developed. Three kinds of social pair interactions are distinguished: Avoidance processes, compromising processes, and imitative processes. Computational results are presented for a special case of imitative processes: the competition of two equivalent strategies. They show a phase transition that describes the selforganization of a behavioral convention. This phase transition is further analyzed by examining the equations for the most probable behavioral distribution, which are BOLTZMANN-like equations. Special cases of BOLTZMANN-like equations do not obey the H-theorem and have oscillatory or even chaotic solutions. A suitable TAYLOR approximation leads to the socalled game dynamical equations (also known as selection-mutation equations in the theory of evolution).

1 Introduction

It is well-known that MARKOVian stochastic processes can be described by a *master equation*. The master equation has found many applications in thermodynamics [1], chemical kinetics [2], laser theory [3] and biology [4]. Moreover, in the last decade WEIDLICH and HAAG have successfully introduced it for the description of social processes [5, 6] like opinion formation [7], migration [8], agglomeration [9] and settlement processes [10].

Since the master equation is difficult to solve (even numerically) one often examines the equations for the most probable distribution of states, instead. These equations are found to be "BOLTZMANN-*like*" *equations*, and have many applications to the kinetics of gases [11] or chemical reactions [12]. Special cases of BOLTZMANN-like equations have also become increasingly important in quantitative social science, namely the *logistic equation* for the description of limited growth processes [13, 14] and the socalled *gravity model* for intercity migration processes [15]. Recently, BOLTZMANN-like models have been suggested for avoidance processes of pedestrians [16, 17], and for attitude formation by direct pair interactions of individuals occuring in discussions [16, 18]. The models for attitude formation include cases of *oscillatory* or even *chaotic* behavior (see sect. 5.1).

Such behavior is, for example, known from fashion or economics (economic cycles, stock market).

In the following, a master equation for behavioral changes by spontaneous transitions and pair interactions will be developed. It allows the description of the selforganization of behavioral conventions. Three kinds of social pair interactions are distinguished: Imitative processes, avoidance processes, and compromising processes. It turns out that for a special case of imitative processes the *game dynamical equations* result, which are used for the description of cooperation and competition processes.

The game dynamical equations are empiricaly validated [19, 20], and have many important applications in social sciences [21, 22] and economy [23]. They are also a powerful tool in evolutionary biology [24, 25, 26, 27]. Moreover, the LOTKA-VOLTERRA-*equations* [28, 29] for the description of predator-prey systems in ecology [30] are mathematically equivalent to a special class of game dynamical equations [31].

2 The configurational master equation

Suppose we have a system with a large number $N \gg 1$ of subsystems (e.g. a gas with N atoms). These subsystems are distributed over several states $\boldsymbol{x}$ (which e.g. distinguish the places $\boldsymbol{r}$ and velocities $\boldsymbol{v}$ of the atoms). If the *occupation number* n_x means the number of subsystems that are in state $\boldsymbol{x}$, we have the relation

$$\sum_x n_x = N \,. \tag{1}$$

The vector

$$\boldsymbol{n} := (\dots, n_x, \dots)^{\mathrm{tr}} \tag{2}$$

consisting of the occupation numbers is called the *configuration* of the system (since it contains all information about the distribution of the N subsystems over the states $\boldsymbol{x}$). $P(\boldsymbol{n}, t)$ shall denote the probability to find the configuration $\boldsymbol{n}$ at time t. This implies

$$0 \le P(\boldsymbol{n}, t) \le 1 \qquad \text{and} \qquad \sum_n P(\boldsymbol{n}, t) = 1 \,. \tag{3}$$

The temporal development of the probability $P(\boldsymbol{n}, t)$ is governed by a *master equation* [32]:

$$\begin{aligned} \frac{d}{dt} P(\boldsymbol{n}, t) &= \text{inflow into } \boldsymbol{n} \qquad\qquad - \text{ outflow from } \boldsymbol{n} \\ &= \sum_{n'} w(\boldsymbol{n}|\boldsymbol{n}'; t) P(\boldsymbol{n}', t) - \sum_{n'} w(\boldsymbol{n}'|\boldsymbol{n}; t) P(\boldsymbol{n}, t) \,. \end{aligned} \tag{4}$$

$w(\boldsymbol{n}'|\boldsymbol{n}; t)$ are the *configurational transition rates* of transitions from configuration $\boldsymbol{n}$ to configuration $\boldsymbol{n}'$. Often the dynamics of the system is mainly given by *spontaneous transitions* and *direct pair interactions* of subsystems. In this case, the configurational transition rates are of the following form [32]

$$w(\boldsymbol{n}'|\boldsymbol{n}; t) := \begin{cases} w_1(\boldsymbol{x}'|\boldsymbol{x}; t) n_x & \text{if } \boldsymbol{n}' = \boldsymbol{n}_{x'x} \\ w_2(\boldsymbol{x}', \boldsymbol{y}'|\boldsymbol{x}, \boldsymbol{y}; t) n_x n_y & \text{if } \boldsymbol{n}' = \boldsymbol{n}_{x'y'xy} \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

- Spontaneous changes of the state from $\boldsymbol{x}$ to $\boldsymbol{x}'$ with an *individual* transition rate $w_1(\boldsymbol{x}'|\boldsymbol{x};t)$ correspond to transitions of the configuration from $\boldsymbol{n}$ to

$$\boldsymbol{n}_{x'x} := (\dots,(n_{x'}+1)\dots,(n_x-1),\dots)^{\mathrm{tr}} \tag{6}$$

with a *configurational* transition rate $w(\boldsymbol{n}_{x'x}|\boldsymbol{n};t) = w_1(\boldsymbol{x}'|\boldsymbol{x};t)n_x$, which is proportional to the number n_x of subsystems that can change the state $\boldsymbol{x}$.

- Pair interactions leading one subsystem to change the state from $\boldsymbol{x}$ to $\boldsymbol{x}'$ and another subsystem to change the state from $\boldsymbol{y}$ to $\boldsymbol{y}'$ correspond to a transition of the configuration from $\boldsymbol{n}$ to

$$\boldsymbol{n}_{x'y'xy} := (\dots,(n_{x'}+1),\dots,(n_x-1),\dots,(n_{y'}+1),\dots,(n_y-1),\dots)^{\mathrm{tr}} \tag{7}$$

with a configurational transition rate $w(\boldsymbol{n}_{x'y'xy}|\boldsymbol{n};t) = w_2(\boldsymbol{x}',\boldsymbol{y}'|\boldsymbol{x},\boldsymbol{y};t)n_x n_y$, which is proportional to the number $n_x n_y$ of possible pair interactions between subsystems that are in state $\boldsymbol{x}$ resp. $\boldsymbol{y}$ (if $n_x \gg 1$ where $P(\boldsymbol{n},t)$ is not negligible, see [32]).

The description of social processes often requires *generalized* configurational transition rates of the form

$$w(\boldsymbol{n}'|\boldsymbol{n};t) := \begin{cases} w_1(\boldsymbol{x}'|\boldsymbol{x};\boldsymbol{n};t)n_x & \text{if } \boldsymbol{n}' = \boldsymbol{n}_{x'x} \\ w_2(\boldsymbol{x}',\boldsymbol{y}'|\boldsymbol{x},\boldsymbol{y};\boldsymbol{n};t)n_x n_y & \text{if } \boldsymbol{n}' = \boldsymbol{n}_{x'y'xy} \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

since individuals may react on the actual (socio)configuration $\boldsymbol{n}$. The dependence of the *individual* transition rates w_1 and w_2 on $\boldsymbol{n}$ reflects *indirect interactions* of the individuals.

3 Equations for behavioral changes

For the description of a system of N individuals, the states $x \in \{1,\dots,S\}$ shall represent the possible *behavioral strategies* of individuals concerning a certain situation. The pair interactions

$$x',y' \longleftarrow x,y\,, \tag{9}$$

during which the strategies are changed from x and y to x' and y', can be completely classified according to the following scheme:

$$\left.\begin{array}{rcl} x,x & \longleftarrow & x,x \\ x,y & \longleftarrow & x,y \end{array}\right\}(0) \tag{10}$$

$$\left.\begin{array}{rcll} x,x & \longleftarrow & x,y & (x\neq y) \\ y,y & \longleftarrow & x,y & (x\neq y) \end{array}\right\}(\mathrm{I}) \tag{11}$$

$$\left.\begin{array}{rcll} x,y' & \longleftarrow & x,x & (y'\neq x) \\ x',y & \longleftarrow & y,y & (x'\neq y) \\ x',y' & \longleftarrow & x,x & (x'\neq x, y'\neq x) \end{array}\right\}(\mathrm{II}) \tag{12}$$

$$\left.\begin{array}{rcll} x,y' & \longleftarrow & x,y & (x\neq y, y'\neq y, y'\neq x) \\ x',y & \longleftarrow & x,y & (x\neq y, x'\neq x, x'\neq y) \\ x',y' & \longleftarrow & x,y & (x\neq y, x'\neq x, y'\neq y, x'\neq y, y'\neq x) \end{array}\right\}(\mathrm{III}) \tag{13}$$

$$
\left.\begin{array}{rcll}
y, x & \longleftarrow & x, y & (x \neq y) \\
x', x & \longleftarrow & x, y & (x \neq y, x' \neq x, x' \neq y) \\
y, y' & \longleftarrow & x, y & (x \neq y, y' \neq y, y' \neq x)
\end{array}\right\} (\mathrm{IV}) \tag{14}
$$

Obviously, the interpretation of the above *kinds* $k \in \{0, \mathrm{I}, \ldots, \mathrm{IV}\}$ of pair interactions is the following:

(0) During interactions of kind (0) both individuals do not change their strategy. These interactions can be omitted in the following, since they have no contribution to the change of $P(\boldsymbol{n}, t)$.

(I) The interactions (I) describe *imitative processes* (processes of persuasion), i.e., the tendency to take over the strategy of another individual.

(II) The interactions (II) describe *avoidance processes*, where an individual changes the strategy when meeting another individual using the same strategy. (Processes of this kind are known as aversive behavior, defiant behavior or snob effect.)

(III) The interactions (III) represent some kind of *compromising processes*, where an individual changes the strategy to a new one (the "compromise") when meeting an individual with another strategy. (Such processes are found, if a certain strategy cannot be maintained when confronted with another strategy.)

(IV) The interactions (IV) describe imitative processes, in which an individual changes the strategy despite of the fact, that he or she convinces the interaction partner of his resp. her strategy. Social processes of this kind are very improbable and can normally be neglected.

The different kinds of pair interactions have been discussed in [16, 18, 33]. In the following, our considerations are restricted to imitative processes. The corresponding individual transition rates have, then, the following general form:

$$
\begin{aligned}
w_2(x', y'|x, y; \boldsymbol{n}; t) &= w_2^*(x|y; \boldsymbol{n}; t)\delta_{xx'}\delta_{xy'}(1 - \delta_{xy}) \\
&+ w_2^*(y|x; \boldsymbol{n}; t)\delta_{yy'}\delta_{yx'}(1 - \delta_{xy}) \,.
\end{aligned} \tag{15}
$$

$w_2^*(y|x; \boldsymbol{n}; t)$ is the rate of imitative strategy changes from x to y and shall be specified now: Let $A_{xx'}$ be the *success* of strategy x when confronted with strategy x'. Then,

$$
E(x; \boldsymbol{n}; t) := \sum_{x'} A_{xx'} \frac{n_{x'}(t)}{N} \tag{16}
$$

is the *expected success* of strategy x in interactions with other strategies. With

$$
w_2^*(y|x; \boldsymbol{n}; t) := \frac{\exp\Big[E(y; \boldsymbol{n}; t) - E(x; \boldsymbol{n}; t)\Big]}{D(y, x; t)} \,, \tag{17}
$$

imitative strategy changes from x to y will occur the more frequent, the greater the expected increase

$$
\Delta_{yx} E := E(y; \boldsymbol{n}; t) - E(x; \boldsymbol{n}; t) \tag{18}
$$

of success is, and the smaller the *incompatibility ("distance")*

$$D(y, x; t) \equiv D(x, y; t) > 0 \tag{19}$$

between the strategies x and y is. (17) is a variant of the *multinomial logit model* [16, 34], which has shown to be suitable for the description of decision processes. The ansatz (17) can also be derived by entropy maximization [16] or with the FECHNERian law of psychophysics [16, 35].

4 Selforganization of behavioral conventions by competition between strategies

As an example for the behavioral equations, we shall consider a case where the individuals can choose between two *equivalent* strategies $x \in \{1, 2\}$, i.e., the *payoff matrix* $\underline{A}$ shall be symmetrical:

$$\underline{A} \equiv \left(A_{xx'}\right) := \begin{pmatrix} A+B & B \\ B & A+B \end{pmatrix} . \tag{20}$$

For spontaneous strategy changes we shall assume the simplest form of transition rates:

$$w_1(x'|x; \boldsymbol{n}; t) \equiv W . \tag{21}$$

A situation of the above kind is the avoidance behavior of pedestrians [16, 36]: In pedestrian crowds with two opposite directions of movement, the pedestrians have sometimes to avoid each other in order to exclude a collision. For an avoidance maneuver to be successful, both pedestrians concerned have to pass the respective other pedestrian either on the right hand side ($x = 1$) or on the left hand side ($x = 2$). Otherwise, both pedestrians have to stop. Therefore, both strategies (to pass pedestrians on the right hand side or to pass them on the left hand side) are equivalent, but the success of a strategy grows with the number n_x of individuals who use the *same* strategy. In the payoff matrix (20) we have $A > 0$, then.

Empirically one finds that the probability $P(1)$ for choosing the right hand side is usually different from the probability $P(2) = 1 - P(1)$ for choosing the left hand side (see fig. 1a). As a consequence, opposite directions of motion normally use separate lanes (see fig. 1b).

We will now examine, if the behavioral model can explain this *break of symmetry*. Figure 2 shows some computational results for $D(y, x; t) \equiv 2$ and $A = 1$. If

$$\kappa := 1 - 4W < 0 , \tag{22}$$

the configurational distribution is unimodal and symmetrical with respect to $n_1 = N/2 = n_2$, i.e., both strategies will be chosen by about one half of the individuals. At the *critical point* $\kappa = 0$ there appears a *phase transition*. This is indicated by the broadness of the probability distribution $P(\boldsymbol{n}, t) \equiv P(n_1, n_2; t) = P(n_1, N - n_1; t)$, which is due to *critical fluctuations*. For $\kappa > 0$ the configurational distribution becomes bimodal in the course of time, so that one of the two equivalent strategies will very probably be chosen by a majority of individuals. This can be interpreted as *selforganization of a behavioral convention*. Behavioral conventions often obtain a law-like character after some time.

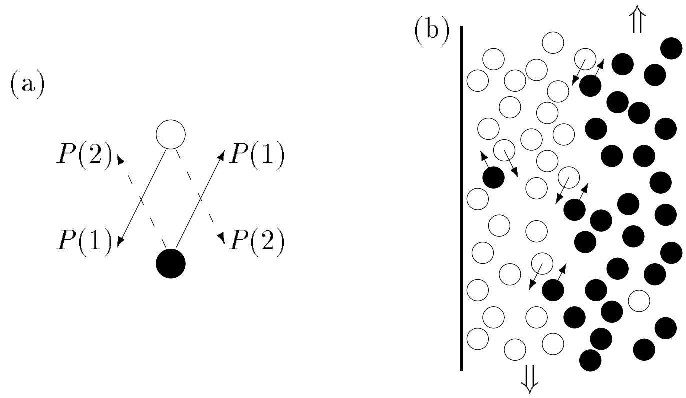

Figure 1: (a) For pedestrians with an opposite direction of motion it is advantageous, if both prefer either the right hand side or the left hand side when trying to pass each other. Otherwise, they would have to stop in order to avoid a collision.
(b) Opposite directions of motion normally use separate lanes. Avoidance maneuvers are indicated by arrows.

5 The most probable strategy distribution

In order to understand the phase transition more explicitly, we shall in the following consider the equations for the *most probable strategy distribution*

$$P(x,t) := \frac{\widehat{n}_x(t)}{N} \tag{23}$$

with

$$P(x,t) \geq 0 \qquad \text{and} \qquad \sum_x P(x,t) = 1\,, \tag{24}$$

where $\widehat{\boldsymbol{n}}(t)$ denotes the most probable (socio)configuration. These equations are approximately given by

$$\frac{d}{dt}P(x,t) = m_x(\widehat{\boldsymbol{n}},t)\,, \tag{25}$$

as can be seen by reformulating the master equation (4) in terms of a LANGEVIN equation [16, 33]. Here,

$$m_x(\widehat{\boldsymbol{n}},t) := \sum_{x'} \Big[\overline{w}(x|x';\widehat{\boldsymbol{n}};t)P(x',t) - \overline{w}(x'|x;\widehat{\boldsymbol{n}};t)P(x,t)\Big] \tag{26}$$

are *drift coefficients*, and

$$\overline{w}(x'|x;\widehat{\boldsymbol{n}};t) := w_1(x'|x;\widehat{\boldsymbol{n}};t) + \sum_{y'}\sum_{y} w_2(x',y'|x,y;\widehat{\boldsymbol{n}};t)\widehat{n}_y \tag{27}$$

have the meaning of *effective transition rates* [32]. It turns out that the explicit equations for the most probable strategy distribution $P(x,t)$ are BOLTZMANN-like equations:

$$\begin{aligned}
\frac{d}{dt}P(x,t) &= \sum_{x'} \Big[\widehat{w}_1(x|x';t)P(x',t) - \widehat{w}_1(x'|x;t)P(x,t)\Big] && \text{(28a)}\\
&+ \sum_{x'}\sum_{y}\sum_{y'} \widehat{w}_2(x,y'|x',y;t)P(x',t)P(y,t) \\
&- \sum_{x'}\sum_{y}\sum_{y'} \widehat{w}_2(x',y'|x,y;t)P(x,t)P(y,t) && \text{(28b)}
\end{aligned}$$

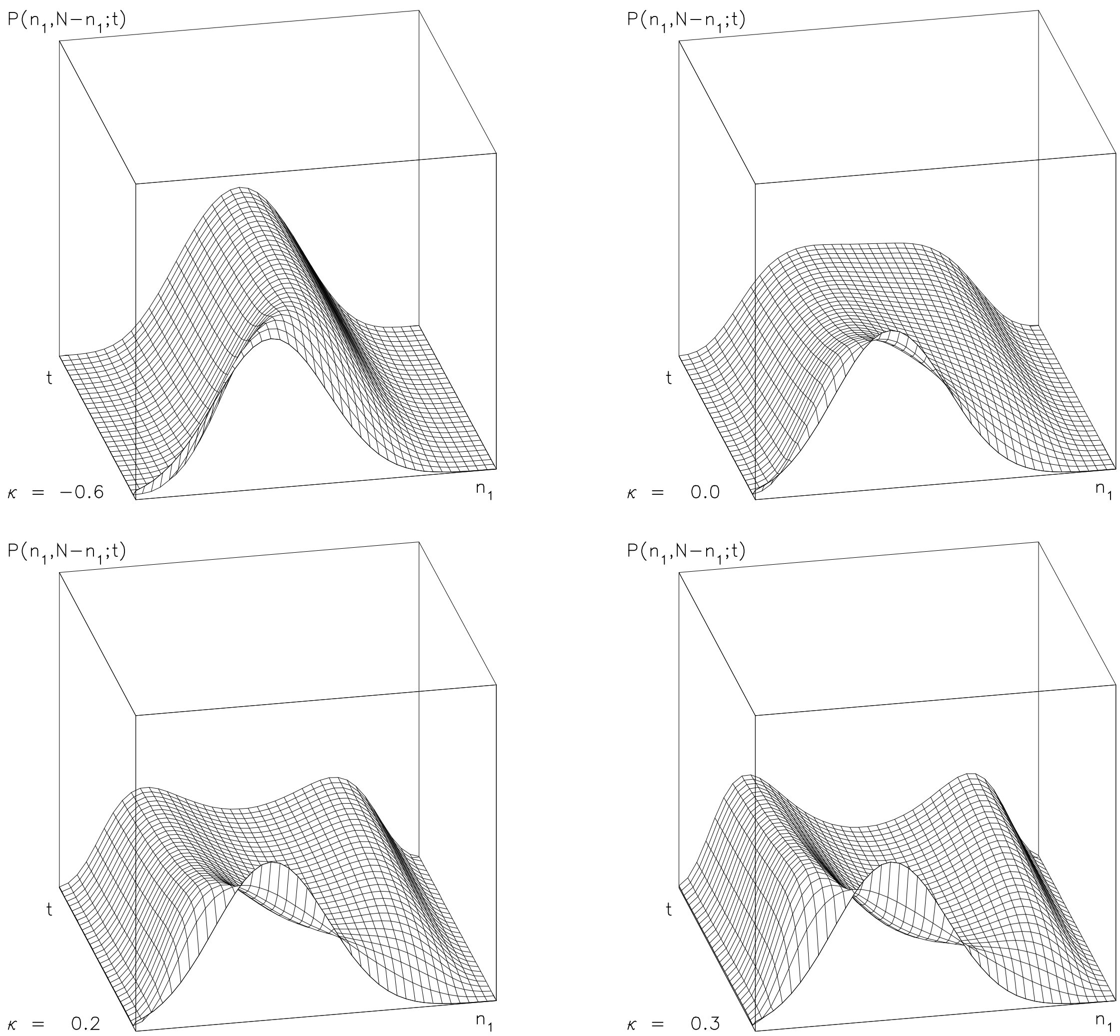

Figure 2: Probability distribution $P(\boldsymbol{n}, t) \equiv P(n_1, N - n_1; t)$ of the (socio)configuration $\boldsymbol{n}$ for varying values of the control parameter κ. For $\kappa = 0$ a phase transition occurs: Whilst for $\kappa < 0$ both strategies are used by about one half of the individuals, for $\kappa > 0$ very probably one of the strategies will be prefered after some time. That means, a behavioral convention develops by social selforganization.

with

$$\widehat{w}_1(x'|x;t) := w_1(x'|x;\widehat{\boldsymbol{n}};t)\,, \tag{29}$$

$$\widehat{w}_2(x',y'|x,y;t) := N w_2(x',y'|x,y;\widehat{\boldsymbol{n}};t)\,. \tag{30}$$

Obviously, the terms (28b) are BOLTZMANN (collision) terms resulting from pair interactions, whereas the terms (28a) are due to spontaneous transitions.

5.1 Oscillatory and chaotic behavior

For $w_1(x'|x;\widehat{\boldsymbol{n}};t) \equiv 0$, $w_2(x',y'|x,y;\widehat{\boldsymbol{n}};t) \equiv w_2(x',y'|x,y)$, and

$$\sum_{x'}\sum_{y'} w_2(x,y|x',y') = \sum_{x'}\sum_{y'} w_2(x',y'|x,y) \tag{31}$$

equation (28) obeys the famous BOLTZMANN H-theorem [16]

$$\frac{dH}{dt} \le 0 \qquad \text{with} \qquad H(t) := \sum_x P(x,t)\ln P(x,t)\,. \tag{32}$$

According to the H-theorem $P(x,t)$ approaches a stationary solution $P_0(x)$ in the course of time. For example, in a dilute gase the velocity distribution approaches a MAXWELL distribution. However, for social processes the relation (31) may be invalid (since there are no collisional invariants). As a consequence, the corresponding BOLTZMANN-like equations can show oscillatory or even chaotic solutions [16, 33] (see figures 3 and 4).

For example, the special BOLTZMANN equations

$$\frac{d}{dt}P(x,t) = \nu P(x,t)\Big[P(x-1,t) - P(x+1,t)\Big] \qquad \text{with} \qquad x \equiv x \bmod S \tag{33}$$

display *nonlinear oscillations* (see fig. 3): A linear stability analysis around the stationary point $\boldsymbol{P}_0 := (1/S,\dots,1/S)^{\text{tr}}$ shows that the eigenvalues of the corresponding JACOBIan matrix are purely imaginary [16, 18]. Due to the relations

$$\sum_{x=1}^{S} P(x,t) = 1 \qquad \text{and} \qquad \prod_{x=1}^{S} P(x,t) = \textit{const.} \tag{34}$$

the trajectory $\boldsymbol{P}(t) \equiv (P(1,t),\dots,P(S,t))^{\text{tr}}$ moves on a $(S-2)$-dimensional hypersurface. For $S = 3$ strategies the shape of the resulting cycles can be calculated explicitly. It is given by

$$P(2,t) = \frac{1-P(1,t)}{2} \pm \sqrt{\left[\frac{1-P(1,t)}{2}\right]^2 - \frac{C}{P(1,t)}} \tag{35}$$

with

$$P(3,t) = 1 - P(1,t) - P(2,t) \qquad \text{and} \qquad C := P(1,t_0)P(2,t_0)P(3,t_0)\,. \tag{36}$$

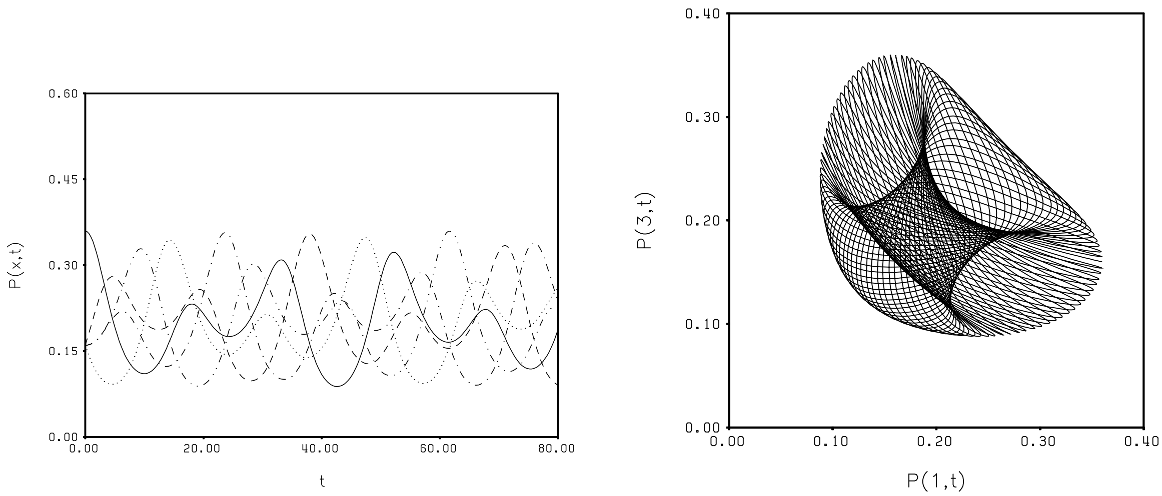

Figure 3: Oscillations are one possible effect of imitative processes. For $S = 5$ different strategies, the oscillatory changes look quite irregular without a short-term periodicity. The corresponding phase portrait has the shape of a torus, which indicates a long-term periodicity by the closeness of the curve.

BOLTZMANN equations with chaotic solutions are, for example, given by the following interaction schemes:

$$\begin{aligned}
4,9 &\xrightarrow{k_1} 1,9\,, \\
1,9 &\xrightarrow{k_1'} 4,9\,, \\
1,9 &\xrightarrow{k_2} 2,9\,, \\
4,6 &\xrightarrow{k_3} 1,6\,, \\
1,1 &\xrightarrow{k_4} 3,3\,, \\
3,9 &\xrightarrow{k_4'} 1,9\,, \\
2,3 &\xrightarrow{k_4''} 1,3\,, \\
8,6 &\xrightarrow{k_5} 6,6\,, \\
6,7 &\xrightarrow{k_5'} 7,7\,, \\
7,8 &\xrightarrow{k_5''} 8,8\,, \\
4,4 &\xrightarrow{k_6} 4,5\,, \\
5,4 &\xrightarrow{k_6'} 4,4\,,
\end{aligned} \tag{37}$$

where k_l denote the interaction rates $\widehat{w}_2(x',y'|x,y)$ of the pair interactions

$$x,y \xrightarrow{k_l} x',y'\,. \tag{38}$$

Using the abbreviations

$$\begin{array}{rclrclrcl}
\alpha &:=& \sqrt{k_1'k_4'/(2k_4k_4'')}\,, & \beta &:=& k_1'/k_4''\,, & \gamma &:=& \alpha k_6/k_6'\,, \\
\tau(t) &:=& k_1'P(9,0)t\,, & a &:=& k_1/k_1'\,, & b &:=& k_2/k_1'\,, \\
c &:=& \alpha k_5/k_1'\,, & c' &:=& \alpha k_5'/k_1'\,, & c'' &:=& \alpha k_5''/k_1'\,, \\
d &:=& \alpha k_6/k_1'\,, & e &:=& k_6/k_6'\,, & \kappa &:=& \alpha k_3/k_1'\,, \\
\epsilon &:=& 2\alpha k_4/k_1'\,, & \epsilon' &:=& k_1'/(k_4''\alpha)\,. & & &
\end{array} \tag{39}$$

and the scaled variables $y_x(\tau)$ according to

$$P(x,t) =: y_x(\tau) P(9,0) \cdot \begin{cases} \alpha & \text{if } x \in \{1,2,4,6,7,8\} \\ \beta & \text{if } x = 3 \\ \gamma & \text{if } x = 5 \\ 1 & \text{if } x = 9\,, \end{cases} \tag{40}$$

the corresponding BOLTZMANN equations are:

$$\begin{aligned}
\frac{d}{d\tau} y_1(\tau) &= a y_4(\tau) y_9(\tau) - (b+1) y_1(\tau) y_9(\tau) + \kappa y_4(\tau) y_6(\tau) \\
&- \epsilon \Big[[y_1(\tau)]^2 - y_3(\tau) y_9(\tau) \Big] + y_2(\tau) y_3(\tau)\,, \\
\frac{d}{d\tau} y_2(\tau) &= b y_1(\tau) y_9(\tau) - y_2(\tau) y_3(\tau)\,, \\
\epsilon' \frac{d}{d\tau} y_3(\tau) &= \epsilon \Big[[y_1(\tau)]^2 - y_3(\tau) y_9(\tau) \Big]\,,
\end{aligned}$$

$$\begin{aligned}
\frac{d}{d\tau} y_4(\tau) &= -a y_4(\tau) y_9(\tau) + y_1(\tau) y_9(\tau) - \kappa y_4(\tau) y_6(\tau) \\
&- d y_4(\tau) \Big[y_4(\tau) - y_5(\tau) \Big]\,, \\
e \frac{d}{d\tau} y_5(\tau) &= d y_4(\tau) \Big[y_4(\tau) - y_5(\tau) \Big]\,, \\
\frac{d}{d\tau} y_6(\tau) &= y_6(\tau) \Big[c y_8(\tau) - c' y_7(\tau) \Big]\,, \\
\frac{d}{d\tau} y_7(\tau) &= y_7(\tau) \Big[c' y_6(\tau) - c'' y_8(\tau) \Big]\,, \\
\frac{d}{d\tau} y_8(\tau) &= y_8(\tau) \Big[c'' y_7(\tau) - c y_6(\tau) \Big]\,, \\
\frac{d}{d\tau} y_9(\tau) &= 0\,.
\end{aligned} \tag{41}$$

For certain sets of parameters these equations have *chaotic* solutions. Especially, for the parameters

$$\begin{array}{lllllllll}
a & := & 0\,, & b & := & 1.2\,, & & & \\
c & := & 0.46\,, & c' & := & 0.46\,, & c'' & := & 0.46\,, \\
d & := & 100\,, & e & := & 10000\,, & \kappa & = & \textit{varying}\,, \\
\epsilon & := & 0.01\,, & \epsilon' & := & 0.0001 & & &
\end{array} \tag{42}$$

and the initial conditions

$$\begin{array}{llllll}
y_1(0) & := & 0.6\,, & y_2(0) & := & 1.8\,, \\
y_3(0) & := & 0.36\,, & y_4(0) & := & 1\,, \\
y_5(0) & := & 1\,, & y_6(0) & := & 1.12\,, \\
y_7(0) & := & 1 + 0.12 \sin(2\pi/3)\,, & y_8(0) & := & 1 + 0.12 \sin(4\pi/3)
\end{array} \tag{43}$$

several *period doubling sequences* are found, if $\kappa \in [0.15, 0.55]$ is varied (see [16] for a more detailled discussion). Figure 4 shows computational results for $\kappa = 0.32$.

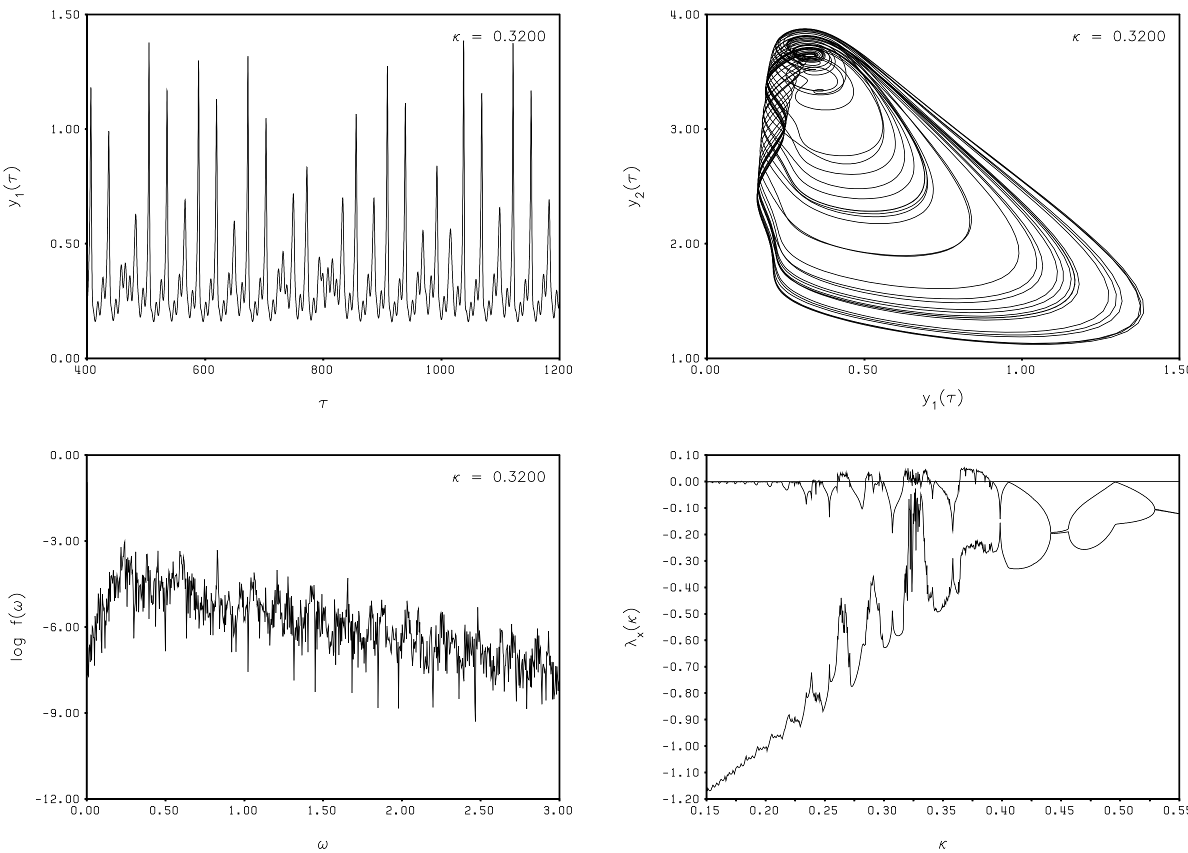

Figure 4: Temporal development, phase portrait, power spectrum $f(\omega)$ and greatest LYAPUNOV exponents $\lambda_x(\kappa)$ of the scaled variables $y_x(\tau) = \mu_x P(x, \nu t)$ for special BOLTZMANN equations that produces chaos.

6 The game dynamical equations

The socalled *game dynamical equations*, which are used for the description of cooperation and competition processes, are also a special case of the BOLTZMANN-like equations. This is shown in the following: If we again restrict our considerations to imitative processes (see (15)), the special BOLTZMANN-like equations

$$\begin{aligned} \frac{d}{dt}P(x,t) &= \sum_{x'} \Big[\widehat{w}_1(x|x';t)P(x',t) - \widehat{w}_1(x'|x;t)P(x,t)\Big] \\ &+ P(x,t)\sum_{x'} \Big[\widehat{w}_2^*(x|x';t) - \widehat{w}_2^*(x'|x;t)\Big]P(x',t) \end{aligned} \tag{44}$$

result with

$$\widehat{w}_2^*(x'|x;t) := N w_2^*(x'|x;\widehat{\boldsymbol{n}};t)\,. \tag{45}$$

Inserting the multinomial logit ansatz (17) and using a suitable TAYLOR approximation leads to the *game dynamical equations*

$$\frac{d}{dt}P(x,t) = \sum_{x'} \Big[\widehat{w}_1(x|x';t)P(x',t) - \widehat{w}_1(x'|x;t)P(x,t)\Big] \tag{46a}$$

$$+ P(x,t)\Big[\widehat{E}(x,t) - \langle \widehat{E}\rangle\Big]\,, \tag{46b}$$

where

$$\widehat{E}(x,t) := E(x;\widehat{\boldsymbol{n}};t) = \sum_{x'} A_{xx'}\frac{\widehat{n}_{x'}(t)}{N} \tag{47}$$

is the *expected success* of strategy x, and

$$\langle \widehat{E}\rangle := \sum_{x'} \widehat{E}(x',t)P(x',t) \tag{48}$$

is the *mean expected success*. Since (46b) can be understood as effect of a *selection* (of strategies with an expected success that exceeds the average $\langle \widehat{E}\rangle$) and (46a) can be interpreted as effect of spontaneous strategy changes (e.g. due to accidental *mutations*) the game dynamical equations are also known as *selection mutation equations* [31, 37]. The mutation term can be used for the description of *trial and error*.

As an example, we shall again examine the case of two equivalent strategies. The game dynamical equations (46) corresponding to (20), (21) have, then, the explicit form

$$\frac{d}{dt}P(x,t) = -2\left(P(x,t) - \frac{1}{2}\right)\Big[W + AP(x,t)\Big(P(x,t) - 1\Big)\Big]\,. \tag{49}$$

According to (49), $P(x) = 1/2$ is a stationary solution. This solution is stable for

$$\kappa := 1 - \frac{4W}{A} < 0\,, \tag{50}$$

i.e., if spontaneous strategy changes are dominating and, therefore, prevent a selforganization process.

At the *critical point* $\kappa = 0$ there appears a *break of symmetry*: For $\kappa > 0$ the stationary solution $P(x) = 1/2$ is unstable, and the game dynamical equations (49) can be rewritten in the form

$$\frac{d}{dt}P(x,t) = -2\left(P(x,t) - \frac{1}{2}\right)\left(P(x,t) - \frac{1+\sqrt{\kappa}}{2}\right)\left(P(x,t) - \frac{1-\sqrt{\kappa}}{2}\right) . \tag{51}$$

That means, for $\kappa > 0$ we have two additional stationary solutions $P(x) = (1+\sqrt{\kappa})/2$ and $P(x) = (1-\sqrt{\kappa})/2$, which are stable. Depending on initial fluctuations, one strategy will win a majority of $100 \cdot \sqrt{\kappa}$ percent. This majority is the greater, the smaller the rate W of spontaneous strategy changes is.

The game dynamical equations (including generalizations and other derivations) are more explicitly discussed in [16, 33].

7 Summary and Conclusions

The master equation and Boltzmann-like equations have shown to be suitable for the quantitative description of behavioral changes and social processes. In the models developed spontaneous strategy changes and behavioral changes due to pair interactions have been taken into account. Three kinds of pair interactions have been distinguished: imitative, avoidance and compromising processes. The game dynamical equations result for a special case of imitative processes. They can be interpreted as equations for the most probable behavioral distribution and allow the description of social selforganization of behavioral conventions.

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